

F-Noetherian Rings and Skew Quantum Ring Extensions

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Dedicated to George Bergman and Donald Passman

Abstract

A ring R shall be called F-noetherian if every finite subset of R is contained in a (left and right) noetherian subring of R . F-noetherian rings have many interesting linear algebra properties which we refer to as the full strong rank condition, fully stably finite, and more generally the basic condition. We also study some basic ring-theoretic properties of F-noetherian rings such as localizations of F-noetherian rings. The F-noetherian property is preserved under some *skew* quantum ring extensions including some iterated Ore extensions and some quantum almost-normalizing extensions. For example, let $R = S[x_1, \dots, x_n]$ be a finitely generated ring over a subring S such that (1) for $i < j$,

$$x_j x_i - q_{ji} x_i x_j \in S[x_1, \dots, x_{j-1}] + Sx_j, \text{ or}$$

$$x_j x_i - q_{ji} x_i x_j \in Sx_1 + \dots + Sx_n + P_2$$

for some units $q_{ji} \in S$ and P_2 is a certain set of quadratic polynomials (related to the quantum group $\mathcal{O}_q(G)$ where G is a connected complex semisimple algebraic group), (2) for all i , $Sx_i + S = S + x_i S$, and (3) each x_i commutes with a subring A of S such that S is finitely generated as a ring over A . Then if S is noetherian, or F-noetherian, or a direct limit of noetherian rings, so is R .

1 Introduction

In this paper we introduce the new notion of an F-noetherian ring and related notions. A ring R shall be called F-noetherian if every finite subset of R is contained in a (left and right) noetherian subring of R . For example, by the Hilbert Basis Theorem, every commutative ring is tightly F-noetherian in the sense that every finite subset of R generates a noetherian subring of R . Tightly F-noetherian rings are directed F-noetherian in the sense that for every finite subset A of R , there exists a noetherian subring $R(A)$ of R containing A such that if $X \subset Y$ are finite subsets of R , then $R(X) \subset R(Y)$. It turns out that the directed F-noetherian rings are precisely the direct limits of noetherian rings by Proposition 8.3.

An F-noetherian ring R has many interesting linear algebra properties. For example,

- (i) if M^m is embedded in M^n and M is a finitely generated non-zero R -module, then $m \leq n$. More generally, for (right or left) R -modules M and N where M is finitely generated, $M \oplus N$ can not be imbedded in M unless $N = 0$, or equivalently, every monomorphism of a finitely generated R -module M has an essential image. For convenience, this condition will be referred to as the full SRC (strong rank condition)
- (ii) every epimorphism $M \rightarrow M$ of a finitely generated R -module M is an isomorphism. That is, every finitely generated (right or left) R -module M is Hopfian. For convenience, this condition will be referred to as the fully stably finite condition.

The above two linear algebra properties are special cases of the following more general property. Every F-noetherian ring R is **basic** in the sense that if we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of (right or left) R -modules and M is a finitely generated R -module, then f is an isomorphism (and $i(A)$ is essential in M). This property was suggested to us by an exercise on commutative and noetherian rings, in [L 3, Exercise 1.10] which is taken from p. 61 of a book entitled "Commutative Noetherian rings and Krull Rings" by Balcerzyk and Jósefiak. (See section 2).

In section 3, we study some ring-theoretic properties of F-noetherian rings. For example, we have the following.

1.1 Remark. .

- (i) F-noetherian rings and tightly F-noetherian rings are preserved under direct limits and homomorphic images.

- (ii) If R and S are tightly F-noetherian (resp. F-noetherian) rings, then so is $R \times S$.
- (iii) If R is an F-noetherian domain, then R is an Ore domain.
- (iv) Let R be an F-noetherian ring and let S be a multiplicative subset of R . Suppose there exists a finite set Q in R such that $sR' = R's$ for every element $s \in S$ and every subring R' of R containing both S and Q . Then S is a denominator set of R and the localization ring RS^{-1} is F-noetherian.
- (v) If R has a non-negative filtration whose associated graded ring $\text{gr}(R)$ is F-noetherian, then the ring R may fail to be F-noetherian. This failure may even occur if $\text{gr}(R)$ is commutative.

For more details about the above facts, see (3.6)–(3.9) and (5.10).

In section 4, we study the preservation of the F-noetherian property under quantum iterated Ore extensions and Laurent ring extensions. We shall see that if S is an F-noetherian ring, then its differential skew polynomial rings may *fail* to be F-noetherian. However, we have the following.

1.2 Theorem. *Let $R = S[x_1; f_1, d_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ be an iterated Ore extension over a ring S such that*

- (1) *for each $j > i$, $f_j(x_i) = q_{ji}(x_i)$ for some unit $q_{ji} \in S$,*
- (2) *each f_i is the identity on S ,*
- (3) *we have one of the following cases.*

Case 1: *For all i , $d_i(S) \subset S$, and all d_i are commuting locally nilpotent derivations on S .*

Case 2: *S has an ascending chain of subsets A_m with $m \in \mathbb{N}$, such that $A_0 = 0$, S is the union of all A_m , and*

$$d_i(A_m) \subset A_{m-1}$$

for all $m \in \mathbb{N}$, and $i = 1, 2, \dots, n$.

Then,

- (i) *if S is F-noetherian, so is R , and*
- (ii) *if S is tightly F-noetherian, then R is directed F-noetherian.*

Note that we have no restrictions on each $d_j(x_i)$ for $j > i$

Similarly, in the case of skew-Laurent rings, if S is F-noetherian, then $R = S[x, x^{-1}; f]$ may fail to be F-noetherian. However, we have

1.3 Proposition. *Let $R = S[x_1, x_1^{-1}; f_1] \dots [x_n, x_n^{-1}; f_n]$ be an iterated skew-Laurent ring extension such that*

- (1) *for each $j > i, f_j(x_i) = q_{ji}(x_i)$ for some unit $q_{ji} \in S$,*
- (2) *each f_i is the identity on S .*

If S is F-noetherian, then so is R .

In section 5, we study the preservation of the F-noetherian property under many types of skew quantum ring extensions starting with almost centralizing extensions in the sense of Passman in [P]. The first general result in this section is Theorem 1.4 below (stated in Theorem 5.4) in which the noetherian part (i) in case 3 is a simple generalization of Proposition I.8.17 in [B-G, p. 77] which is one of the key steps in proving that the quantum group $\mathcal{O}_q(G)$ is noetherian (where G is a connected complex semisimple algebraic group and q is a fixed non-root of unity). Specifically, Proposition I.8.17 in [B-G] assumes (versus our generalization below) that each x_i commutes with the elements of S , it assumes our $f = 0$ and each $(x_j x_i - q_{ji} x_i x_j) = 0$ if $i = 1$. Finally, the noetherian part (i) in case 2, is a generalization of [L-R, Cor. 2.4] since we do not assume any PBW S -basis of R .

1.4 Theorem. *Let $R = S[x_1, \dots, x_n]$ be a finitely generated ring over a subring S satisfying the following two conditions.*

- (1) *for all i , $x_i S + S = S + S x_i$.*
- (2) *for all $j > i$, there exist units $q_{ji} \in S$ such that we have one of the following cases.*

Case 1: $(x_j x_i - q_{ji} x_i x_j) \in S[x_1, \dots, x_{j-1}] + S x_j$

Case 1': $(x_j x_i - q_{ji} x_i x_j) \in S x_i + S[x_{i+1}, \dots, x_n]$

Case 2: $(x_j x_i - q_{ji} x_i x_j) \in S + S x_1 + \dots + S x_n$

Case 3: $x_j x_i - q_{ji} x_i x_j = f + g$ where $f \in S + S x_1 + \dots + S x_n$ and if $i > 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

Then in all cases, we have the following.

- (i) if S is noetherian, then so is R .
- (ii) Now we suppose that each x_i commutes with the elements of a subring A of S such that S is finitely generated as a ring over A . Then, if S is F -noetherian, so is R .
- (iii) Under the additional assumption in (ii), if S is directed F -noetherian, then so is R .

Then we mix the cases of Theorem 1.4 in two different ways that are stated in Theorems 5.8 and 5.9. For example, Theorem 5.8 says the following.

1.5 Theorem. Let $G = k[t_1, \dots, t_m, x_1, \dots, x_n]$ be a finitely generated ring over its subring k such that

- (i) For all $j > i$, there exist units $p_{ji} \in k$ such that

$$t_j t_i - p_{ji} t_i t_j \in k[t_1, \dots, t_{j-1}] + k t_j$$

- (ii) For all $j > i$, there exist units $q_{ji} \in k$ such that

Case 1: $(x_j x_i - q_{ji} x_i x_j) \in k[x_1, \dots, x_{j-1}] + k x_j$

Case 2: $(x_j x_i - q_{ji} x_i x_j) \in k + k x_1 + \dots + k x_n$

Case 3: $(x_j x_i - q_{ji} x_i x_j) = f + g$ where $f \in S + S x_1 + \dots + S x_n$ and if $i > 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

- (iii) For all j and i , there exist units $c_{ji} \in k$ such that

$$t_i x_j - c_{ji} x_j t_i \in k + k t_1 + \dots + k t_u + k x_1 + \dots + k x_v$$

where in case 1, $(u, v) = (i, j - 1)$ or $(i - 1, j)$; while in case 2, $(u, v) = (i, n)$; and in case 3, $(u, v) = (i - 1, j)$.

Moreover,

- (1) for all i, j , $k t_i + k = k + t_i k$ and $k x_j + k = k + x_j k$, and

- (2) each t_i and each x_i commute with the elements of a subring A of S such that S is finitely generated as a ring over A .

Then in all cases, if k is noetherian or F -noetherian or directed F -noetherian, so is G .

In section 6, we shall give many examples. For example, we generalize the Hayashi example in (6.5) as follows.

1.6 Example. *A generalization of the Hayashi Example.* Let R be the k -algebra generated by the variables x_i, y_i, z_i , $1 \leq i \leq n$ such that any pair of variables almost-commute (in the sense that $ab = q_{ab}ba$ where q_{ab} is a unit in k) except for the pairs (x_i, z_i) where we have the relations

$$(z_i x_i + q x_i z_i) y_i = 1 = y_i (z_i x_i + q x_i z_i)$$

for some unit $q \in S$. Then if k is noetherian or F -noetherian, so is R .

In section 7, we generalize many quantum groups as in the following example stated in (7.2).

1.7 Example. *A generalization of the quantum group $\mathcal{O}_q(M_n(k))$*

Let $G(\mathcal{O}_q(M_n(k))) = k[t_1, \dots, t_m, x_{11}, \dots, x_{nn}]$ be the k -algebra generated by the variables $\{t_1, \dots, t_m, x_{11}, \dots, x_{nn}\}$ such that

- (i) $k[x_{11}, \dots, x_{nn}] = \mathcal{O}_q(M_n(k))$ (with q being a central unit of k) is the k -algebra with the standard relations of the quantum group of $\mathcal{O}_q(M_n(k))$ of $n \times n$ matrices. See [B-K, p. 16].

- (ii) For all $j > i$, there exist units $p_{ji} \in k$ such that

$$\textbf{Case 1: } t_j t_i - p_{ji} t_i t_j \in k[t_1, \dots, t_{j-1}] + k t_j$$

$$\textbf{Case 2: } t_j t_i - p_{ji} t_i t_j \in k + k t_1 + \dots + k t_m$$

- (iii) Let $\{x_1, \dots, x_{n^2}\}$ be the lexicographic ordering of $\{x_{11}, \dots, x_{nn}\}$. For all j and i , we also assume that there exist units c_{ji} in k such that

$$t_i x_j - c_{ji} x_j t_i \in k + k t_1 + \dots + k t_u + k x_1 + \dots + k x_v$$

where in case 1, $(u, v) = (i, j-1)$ or $(i-1, j)$, while in case 2, $(u, v) = (m, n^2)$

If k is noetherian, or F -noetherian, or directed F -noetherian, then so is $G(\mathcal{O}_q(M_n(k)))$.

In section 8, we shall give many examples of F-noetherian matricial rings. For example, we show that $M_2(Z[x])$ is noetherian but not tightly F-noetherian.

We also give many examples of F-noetherian group rings. For example, let $K[G]$ be the group algebra of a locally finite group G over an F-noetherian ring K . (Recall that a group G is locally finite if every finitely generated subgroup is finite). (Interesting examples of G are the finitary symmetric/alternating groups on an infinite set). Or let $K[G]$ be the group algebra of a polycyclic-by-finite group G (for example G is a finitely generated nilpotent group) over an F-noetherian ring K . Then in both cases $K[G]$ is F-noetherian. (See 8.2.4 and 8.2.5)

Finally, in section 9, we pose few open problems. For example,

Problem 1. Find an example of an F-noetherian ring which is not a direct limit of noetherian rings. Or equivalently, find an example of an F-noetherian ring which is not directed F-noetherian.

Throughout the paper, we shall use the notation

$$R = S[x_1, x_2, \dots, x_n]$$

for the ring generated by a subring S and the elements x_1, \dots, x_n of R .

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2 Linear Algebra over Commutative and Noetherian rings

Every ring R has an identity element 1, all subrings of R have 1, all R -modules are unital, and all ring morphisms from R to S take 1_R into 1_S . Moreover, recall that a ring is noetherian if it is left and right noetherian .

First we recall the following definitions in [L 2, Chapter 1].

2.1 Definition. A ring R satisfies the (right) **strong rank condition** (SRC) if, whenever we have a monomorphism of right R -modules $f : R^m \rightarrow R^n$, then $m \leq n$. Equivalently, any set of linearly independent elements in $(R^n)_R$ has cardinality $\leq n$. We shall reserve the term strong rank condition (SRC) for both right SRC and left SRC.

2.2 Definition. A ring R is **stably finite** if every epimorphism of (right or left) R -modules $f : R^n \rightarrow R^n$ is an isomorphism. Equivalently, any generating set of n elements of R^n is a basis of R^n . Or equivalently, the matrix rings $M_n(R)$ are Dedekind-finite, i.e, they satisfy the property: $ab = 1$ implies $ba = 1$.

For the basic properties of such rings, see [L 2, Chapter 1].

2.3 Fact. *Commutative rings and noetherian rings have the SRC*

The Strong Rank Condition (SRC) for commutative rings is a fact that does not seem to be well known as it should be, as remarked in [L 2, p. 15]). Moreover, every (left and right) noetherian ring has the strong rank condition (SRC). This is proved in a simple way in [L 2, p. 14]. For another proof, one can take the (Goldie) uniform dimension in the embedding $R^m \rightarrow R^n$, to get $m \dim(R) \leq n \dim(R)$. Thus $m \leq n$ (since R is a non-zero noetherian ring). The proof in the commutative case can be easily reduced to the noetherian case by the Hilbert Basis Theorem.

2.4. We will see that commutative rings and noetherian rings satisfy the following three further properties, for which we shall give specific names for easy reference.

- (i) **Full SRC.** A ring R has the *full SRC condition* if, for (right or left) R -modules M and N where M is finitely generated, $M \oplus N$ can not be imbedded in M unless $N = 0$. In particular, if M^m is embedded in M^n and M is a finitely generated non-zero R -module, then $m \leq n$. Equivalently, a ring R has the full SRC if every monomorphism of a finitely generated R -module M has an essential image.
- (ii) **Fully stably finite.** A ring R is called *fully stably finite* if every epimorphism $M \rightarrow M$ of a finitely generated R -module M is an isomorphism. That is, every finitely generated (right or left) R -module M is Hopfian.

- (iii) **Basic.** A ring R is called *basic* if it has the following very interesting property. Suppose we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of (right or left) R -modules. If M is a finitely generated R -module, then f is an isomorphism. Or equivalently, by (2.6) below, R is a basic ring if it has the following (seemingly a bit stronger) property. Suppose we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of (right or left) R -modules. If M is a finitely generated R -module, then f is an isomorphism and $i(A)$ is *essential* in M .

Note that the basic property above resembles “the Schröder Bernstein Theorem” for injective R -modules [L 2 or 3, Ex. 3.31]. Observe, in the above setting, that if A is finitely generated (instead), then M is finitely generated via f .

2.5 Comment. The above notion of basic rings was motivated to us by a very interesting exercise (stated below) which has many noteworthy special cases as described in [L 3, Ex. 1.10, p. 7] which is taken from p. 61 of the book of Balcerzyk and Josefiak, “Commutative Noetherian Rings and Krull Rings,” Halsted Press/Polish Sci. Publishers, 1989.

Exercise: Suppose we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of R -modules. Then f is an isomorphism in the following cases.

1. A is a noetherian module
2. R is commutative and M is finitely generated

2.6 Theorem.

- (i) *Basic \implies full SRC and fully stably finite.*
- (ii) *A ring R is basic if it has the following (seemingly stronger) property: Suppose we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of R -modules. If M is a finitely generated R -module, then f is an isomorphism and $i(A)$ is essential in M .*
- (iii) *Commutative rings and noetherian rings are basic.*

Proof.

- (i) Let R be a basic ring. To prove R has the full SRC, let M be a finitely generated R -module suppose $M \oplus A$ is embedded in M . Then consider the projection $f : M \oplus A$ onto M . Since R is basic, the projection map f must be an isomorphism. Hence $A = 0$, so R has the full SRC. Now let $f : M \rightarrow M$ be an epimorphism of a finitely generated R -modules and consider the identity morphism $M \rightarrow M$. Since R is basic, f must be an isomorphism, and R is fully stably finite.
- (ii) Assume R to be basic and suppose we have a monomorphism $i : A \rightarrow M$ and an epimorphism $f : A \rightarrow M$ of (right or left) R -modules where M is a finitely generated R -module. Since R is basic, f is an isomorphism, so A and M are isomorphic. To prove $i(A)$ is essential in M , suppose $i(A) \oplus B$ is embedded in M . But $i(A)$, A , and M are isomorphic. Hence $M \oplus B$ is embedded in M , so $B = 0$ by (i). Hence $i(A)$ is essential in M .
- (iii) Now suppose we have an epimorphism $f : A \rightarrow M$ and a monomorphism $i : A \rightarrow M$ of R -modules where M a finitely generated R -module. If R is commutative, then f is an isomorphism by Exercise (2.5), so R is basic. If R is noetherian, then the finitely generated R -module M is a noetherian, so $i(A)$ and A are noetherian R -modules. Hence Exercise (2.5) applies to obtain that f is an isomorphism. Hence R is basic.

□

Note. After writing the paper, F. Kourki pointed out that rings with the *full SRC* are characterized (without names) in [Hag-V, Thm. 3.3]. Moreover, our *fully stably finite* rings are characterized (without names) via their matrix rings in [G, Thm. 7]. Furthermore, our *basic* rings are precisely the left and right II_1 rings in [D] where such rings are defined as follows: every epimorphism from a submodule of a finitely generated R -module M onto M is an isomorphism. The main result in [D] is that Left II_1 rings are closed under direct limits [D, Thm. 2]. Finally, certain P.I. rings are left II_1 (see [A-F-S, Thm. 2.2]).

3 F-Noetherian Rings

Our modules will be viewed as right modules and the arguments follow similarly for left modules.

First we introduce the following new definition.

3.1 Definition. A ring R is called F-noetherian (for finitely noetherian) if every finite subset of R is contained in a (left and right) noetherian subring of R . In other words, a ring R is an F-noetherian ring if R is a union of (not necessarily directed) noetherian subrings R_i with the additional property that every finite subset of R is contained in some R_i .

3.2 Definition. A ring R is called *directed* F-noetherian if for every finite subset A of R , there exists a noetherian subring $R(A)$ of R containing A such that if $X \subset Y$ are finite subsets of R , then $R(X) \subset R(Y)$.

3.3 Definition. A ring R is called *tightly* F-noetherian if every finite subset of R generates (as a ring) a noetherian subring of R . Note that in this case, R is directed F-noetherian since R is the union of the subrings R_i which are generated by the finite subsets of R .

Hence we have the following trivial implications.

Tightly F -noetherian \implies directed F -noetherian $\implies F$ -noetherian

Moreover, in Proposition (8.3) we will see that

Directed F-noetherian rings \equiv Direct limits of noetherian rings

3.4 Theorem. *F-noetherian rings are basic rings. In particular, F-noetherian rings have the full SRC and the fully stably finite condition.*

Proof. By (2.7) it suffices to prove that an F-noetherian ring R is basic where M is a finitely generated R -module. We must prove that f is an isomorphism.

In case R is commutative, one can easily reduce the proof about f being an isomorphism to the noetherian case verbatim as done in the solution to Exercise (1.4)(ii) in [L 3, Ex. 1.10, p. 7]. We comment that this reduction resembles the reduction to the noetherian case done by Strooker in the mid 60's in proving that, if R is a commutative ring, every epimorphism of R^n is an isomorphism. In fact, this last property was also proved later by

Vasconcelos. However the solution to Exercise (1.4)(ii) in [L 3, p. 7] only requires that every finite subset of R is contained in a noetherian subring of R , which is exactly what we have since R is F-noetherian. Hence f is an isomorphism. \square

3.5 Example. The first examples of F-noetherian rings are commutative rings which are also tightly F-noetherian via the Hilbert Basis Theorem as shown in (2.1). Division rings and more generally noetherian rings are obviously F-noetherian.

3.6 Remark.

- (i) F-noetherian rings and tightly F-noetherian rings are preserved under homomorphic images.
- (ii) F-noetherian rings and tightly F-noetherian rings are preserved under direct limits.
- (iii) A subring (which contains the unity 1) of a tightly F-noetherian ring R is trivially a tightly F-noetherian ring.
- (iv) A subring of an F-noetherian ring may fail to be an F-noetherian ring.

Example. Note that we can not take the field of fractions of a commutative polynomial ring P in infinitely many variables over a field because P is commutative whence (tightly) F-noetherian. So let R be any division hull of the free ring $\mathbb{Z}[a, b]$ on two generators [L 1, (14.25)]. Then R is noetherian being a division ring. However, the subring $S := \mathbb{Z}[a, b]$ is not noetherian, for example, because the uniform dimension of S as a right S -module is infinite as in [L 2, (1.31)]. But S is generated by $\{a, b\}$. Hence S is not F-noetherian.

3.7 Proposition.

- (i) If R and S are F-noetherian rings, then so is $R \times S$.
- (ii) If R and S are tightly F-noetherian rings, then so is $R \times S$.

Proof. (i) let X be a finite subset of $R \times S$. Then X is contained in $A \times B$ where A and B are finite subsets of R and S respectively. Since R is F-noetherian, A is contained in a noetherian subring R' of R . Similarly, B is contained in a noetherian subring S' of S . Now $R' \times S'$ is a noetherian subring containing X . Hence $R \times S$ is F-noetherian.

(ii) This follows by the evident modification of the proof in part (i) and the following exercise. \square

Exercise If A is a subdirect product of two noetherian rings R and S , then A is noetherian. Recall that a subdirect product of rings R and S is a subring of $R \times S$ which projects surjectively onto each factor.

Proof. First we show that S is a noetherian A -module as follows. S can be viewed as an A -module via the projection $g : R \times S \rightarrow S$ by $a.s = g(a)s$. Let W be a sub A -module of S . For all $s \in S$ and $w \in W$, there exists $a \in A$ (since A is a subdirect product of $R \times S$), such that

$$s.w = g(a).w = a.w. \quad (1)$$

So W is an S -module. But S is noetherian. Hence W is a finitely generated S -module. Consequently, W is a finitely generated A -module by equation (1). Similarly R is a noetherian A -module. Hence $R \times S$ is a noetherian A -module, so every ideal W of A (which is an A -submodule of $R \times S$) is a finitely generated A -module. Hence A is noetherian. \square

3.8 Remark. Let R be an F -noetherian domain. Then R is an Ore domain.

Proof. Let aR and bR be non-zero submodules. Then a and b (with 1) are contained in a noetherian subring S . Since S is a noetherian domain, S is an Ore domain [L 2, (10.23)]. Hence aS and bS have non-zero intersection. Consequently, aR and bR have non-zero intersection, so R is a right Ore domain. Similarly, R is a left Ore domain. \square

3.9 Proposition. Let R be an F -noetherian ring and let S be a multiplicative subset of R .

(i) If S is central in R , then the localization ring RS^{-1} is F -noetherian.

(ii) Suppose there exists a finite set Q in R such that

$$sR' = R's$$

for every element $s \in S$ and every subring R' of R containing $S \cup Q$. Then S is a denominator set for R and the localization ring RS^{-1} is F -noetherian.

(iii) Suppose there exists a finite set Q of R such that

$$sR' = R's$$

for every element $s \in S$ and every **finitely generated** subring R' of R containing $S \cup Q$. If R is also tightly F -noetherian, then S is a denominator set of R and the localization ring RS^{-1} is F -noetherian.

Proof.

- (i) Let $X := \{x_1 = \frac{a_1}{s_1}, \dots, x_n = \frac{a_n}{s_n}\}$ be a subset of RS^{-1} . Then $\{a_1, \dots, a_n; s_1, \dots, s_n\}$ is contained in a noetherian subring R' of R . Let $S' = R' \cap S$. Now S' is a denominator set in R' since S is central in R , and R' is noetherian. Hence the localization $R'(S')^{-1}$ is noetherian [G-W, Cor. 9.18]. But X is contained in $R'(S')^{-1}$. Thus X is contained in a noetherian subring of RS^{-1} .
- (iii) For convenience, we shall prove (iii) before (ii). Let S_0 be any finite set of generators for S . Then the proof of part (i) can be applied if we can check that S' is a denominator set of R' and S is a denominator set of R . The fact they are both Ore sets follows from our assumption (*) that $sR' = R's$ for every element $s \in S$. But every Ore set in a noetherian ring is a denominator set [G-W, Prop. 9.9]. Hence S' is a denominator set of R' . Now we prove that S is right reversible in R . Suppose $sr = 0$ where $s \in S$ and $r \in R$. Then $S_0 \cup (r \cup Q)$ **generates** a finitely generated noetherian subring $S(r)$ of R (since R is tightly F-noetherian). Now S is a Ore set of $S(R)$ by (*) and $S(R)$ is noetherian. As above, S is a right denominator set in $S(R)$, and thus S is right reversible in $S(R)$. Since $\{s, r\} \subset S(R)$, we have $sr = 0$ (where $s \in S$ and $r \in S(R)$) implies that there exists $s^* \in S(R)$ such that $rs^* = 0$. Similarly, we prove that S is left reversible in R . Hence S is a denominator set of R as well.
- (ii) For this part, the proof in part (iii) applies except that now $S_0 \cup (r \cup Q)$ is contained in a noetherian subring $S(r)$ of R (since R is only F-noetherian). But that is fine because our hypothesis now is that every element of S normalizes every subring of R . This proves Proposition 3.9.

□

3.10 Corollary. *Let $R = K[x_1, \dots, x_n]$ be a finitely generated ring over a ring K where each x_i commutes with the elements of K . Let S be finitely generated multiplicatively closed subset of R where S is in the centralizer of K . Suppose K contains a finite subset Q such that for all $s \in S$ and for each x_i , $sx_i = q_i(s)x_is$ for some unit $q_i(s)$ in the subring $[Q]$ generated by Q .*

If R is F-noetherian, then S is a denominator set for R and the localization ring RS^{-1} is F-noetherian.

Proof. This follows very easily from part (ii) proved above as follows. Let R' be any subring of R containing both S and Q . Let $s \in S$ and let $r' \in R'$. For illustration, say $sr' = s(kx_1x_3x_2x_1) \in sR'$ for some $k \in K$. Recall our hypotheses that each $sx_i = q_i(s)x_is$ for some $q_i(s)$ in the subring $[Q]$ generated by Q , and that the centralizer of K contains both S and each x_i . Hence $sr' = s(kx_1x_3x_2x_1) = q(kx_1x_3x_2x_1)s = qr's$ for some $q \in [Q]$. Thus $sr' \in R's$ since R' contains Q . This illustrates that $sR' \subset R's$. Similarly $R's \subset sR'$ since each $q_i(s)$ in $sx_i = q_i(s)x_is$ is a unit in $[Q]$. Consequently, $sR' = R's$, so we can apply part (ii) proved above to obtain our result. \square

4 Quantum Iterated Ore Extensions and Skew Laurent Extensions

4.1. Recall that a *skew polynomial ring* (in one variable) $R = S[x; f, d]$ is the ring of polynomials in x with left coefficients in S , i.e. R is the free (left) S -module with basis $\{x^n \mid n = 0, 1, 2, \dots\}$ such that

- (i) $xs = f(s)x + d(s)$ for all $s \in S$,
- (ii) f is an automorphism of S (sending 1 to 1), and
- (iii) d is a (left) f -derivation of S . That is, $d : S \rightarrow S$ is an additive map such that $d(xy) = f(x)d(y) + d(x)y$ for $x, y \in S$.

Such skew polynomial rings $S[x; f, d]$ are also called *Ore extensions* in which f is an automorphism of S (since f is only assumed to be an endomorphism of S for general Ore extensions). If f is the identity, $S[x; f, d]$ is written simply as $S[x; d]$.

Recall that if S is a noetherian ring, then so is the skew polynomial ring $S[x; f, d]$. This theorem is the **Hilbert Basis Theorem** for skew polynomial rings. For more details, see [G-W, p. 13] (or [M-R, Thm. 1.2.9], [B-G, pp. 8-9], [K, p. 19]).

4.2 Proposition.

- (i) Let $R = S[x; d]$ be a differential skew polynomial ring. If S is F -noetherian, then R may fail to be F -noetherian.

- (ii) Let $R = S[x; d]$ be a differential skew polynomial ring. Suppose d is locally nilpotent (i.e, d acts nilpotently on every element s of S), If S is F -noetherian, then R is F -noetherian. Moreover, if S is tightly F -noetherian, then R is directed F -noetherian.

Proof.

- (i) We shall give an example. Let $S = \mathbb{Z}[x_0, x_1, \dots, x_n, \dots]$ be the commutative ring in infinitely many variables. Let $R = S[t; d]$ be the differential Ore extension where d is the S -linear derivation of S such that $d(x_i) = x_{i+1}$. Then R is not noetherian because $R = S[t; d]$ is a free S -module and S is not noetherian.

Now $\{x_0, t\}$ and 1 generate R as a ring. Hence R is not F -noetherian although S is F -noetherian since S is commutative.

- (ii) This is a special case of Theorem 4.3 as shown next.

□

4.3 Theorem. Let $R := S[x_1; f_1, d_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ be an iterated Ore extension over a ring S such that

- (1) for each $j > i, (f_j)(x_i) = q_{ji}(x_i)$ for some unit $q_{ji} \in S$,
- (2) each f_i is the identity on S ,
- (3) each $d_i(S) \subset S$, and each d_i is locally nilpotent on S .

Then

- (i) If S is F -noetherian, then so is R .
- (ii) If S is tightly F -noetherian, then R is directed F -noetherian.

We assume no restrictions on each $d_j(x_i)$ for $j > i$.

Proof.

- (i) First we note that since R is an iterated Ore extension of S , every element of R is an S -sum of the standard PBW basis in all x_i , namely the monomials $(x_1)^{e_1}, x_2^{e_2} \dots (x_n)^{e_n}$.

Given any finite subset of R , we collect carefully all “relevant” constants relative to B in S . Such finitely many constants are contained in a noetherian subring S^* of S . Then we consider the noetherian

subring $R^* = S^*[x_1; f_1, d_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ which will contain the given finite subset of R .

For clarity, first ***we assume that each d_i is zero on S*** . Thus each x_i commutes the elements of S (because each f_i is the identity on S). Now let B be any finite subset of R . Then each $b \in B$ can be written as an S -sum of the above standard PBW basis of R . Let B' be set of all S -coefficients of all $b \in B$ appearing in S . To B' , we add all q_{ji} and their inverses together with

(*) the coefficients appearing in S of all $\{d_j(x_i) \mid \text{for } j > i\}$ in terms of the standard PBW basis of R .

Let B'' be the resulting finite subset of S . Since S is F-noetherian, B'' is contained in a noetherian subring S^* of S . It is easy to check that $R^* = S^*[x_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ is a *well-defined* iterated Ore extension. Finally, this subring of R is noetherian and evidently containing the given finite set B of R .

Case 1: In general: recall that f_i is the identity on S , each $d_i(S) \subset S$, and d_i are commuting locally nilpotent on S . Now the proof in the general case is very similar to the proof in the above special case (with each $d_i = 0$) except that we need to enlarge the finite set B'' to $D(B'')$ which is the *union* of B'' with the following set $((d_1)^{e_1}(d_2)^{e_2} \dots (d_n)^{e_n})(x)$ with $x \in B''$ and each $e_i \in \mathbb{N}$. Note that $D(B'') \subset S$ since each $d_i(S) \subset S$. Moreover, $D(B'')$ is a finite set because the d_i are commuting locally nilpotent derivations on S . Then, as in the special case above, we take any noetherian subring S^* of S containing $D(B'')$. Now each d_i preserves S^* and each f_i fixes the elements of S^* . So we can form the *well-defined* iterated Ore extension $R^* = S^*[x_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ which is a noetherian subring of R containing the given finite set B of R .

Case 2: The proof is very similar to the proof in case 1 but by enlarging B'' to $D(B'')$ which is the *union* of B'' with following set:

$$\{d_{i_1}, d_{i_2} \dots d_{i_k}(x) \mid x \in B'', \text{ and each } i_k \in \{1, 2, \dots, n\}\}.$$

Note our hypotheses in case 2 makes sure that $D(B'')$ is a finite subset of S .

- (ii) By the proof of (i), for each finite set B , we have constructed the corresponding finite set B'' in S such that if $X \subset Y$ are finite subsets

of R , we have $X'' \subset Y''$, whence $D(X'') \subset D(Y'')$. Since S is tightly F-noetherian, $D(X'')$ and $D(Y'')$ generate noetherian subrings $D(X'')^*$ and $D(Y'')^*$ in R . Moreover, we have $D(X'')^* \subset D(Y'')^*$. Hence R is directed F-noetherian. This proves Theorem 4.3.

□

4.4 Corollary. *Let $R = S[x_1; f_1, d_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ be an iterated Ore extension over a ring S such that*

- (1) *for all $j > i, (f_j)(x_i) = q_{ji}(x_i)$ for some unit q_{ji} in S , and*
- (2) *each x_i commutes with the elements of S ,*

Then

- (i) *if S is F-noetherian, then so is R , and*
- (ii) *if S is tightly F-noetherian, then R is directed F-noetherian.*

4.5 Remark. For comparison, we record now Corollary 5.7 of the next section.

Let $R = S[x_1; f_1, d_1][x_2; f_2, d_2] \dots [x_n; f_n, d_n]$ be an iterated Ore extension over a ring S such that

- (1) for all $j > i, (f_j)(x_i) = q_{ji}(x_i)$ for some unit $q_{ji} \in S$, and
- (2) for all $i, x_i S + S = S + S x_i$, and $S = A[z_1, \dots, z_m]$ (a finitely generated ring over a subring A) such that each x_i commutes with the elements of A ,

then, if S is F-noetherian, then so is R .

4.6 Remark.

- (i) If S is F-noetherian, then $R = S[x, x^{-1}; f]$ may fail to be F-noetherian.
- (ii) Let $R = S[x, x^{-1}; f]$ be a skew-Laurent ring extension such that for all $s \in S, f(s) = q_s s$ for some unit $q_s \in S$ fixed under the automorphism f . If S is tightly F-noetherian, then $R = S[x, x^{-1}; f]$ is directed F-noetherian.

Proof.

- (i) Let $S = \mathbb{Z}[a_0, a_1, a_{-1}, a_2, a_{-2}, \dots, a_n, a_{-n}, \dots]$. Let $R = S[x, x^{-1}; f]$ (where f is an automorphism) be the skew-Laurent ring such that $f(a_i) = a_{i+1}$. That is, $x.(a_i) = f(a_i)x$. Moreover, $x^m(a_i) = f^m(a_i)x^i$ for all $m, i \in \mathbb{Z}$. Since $xa_ix^{-1} = a_{i+1}$ and $x^{-1}a_ix = a_{i-1}$, it follows that a_0, x, x^{-1} generate R . But R is not noetherian since R is a free S -module and S is not noetherian. Hence $R = S[x, x^{-1}; f]$ is not F-noetherian while S is F-noetherian being commutative.
- (ii) For each element b of R , let $C(b)$ be the set of all S -coefficients appearing in writing b as a polynomial in x and x^{-1} . Now, for every $s \in C(b)$, $f(s) = q_s s$ for some unit $q_s \in S$ fixed under f . Let $Q(b) = C(b) \cup \{q_s | s \in C(b)\}$ which is a finite subset of S .

Now let B be a finite subset of R . Let $Q(B)$ be the union of all $Q(b)$ defined above as b varies over B . Let S^* be the subring generated by $Q(B)$. Since for all $s \in S(B)$, q_s is fixed under f , and $f^{-1}(s) = (q_s)^{-1}s$, it follows that $f(S^*) \subset S^*$ and $f^{-1}(S^*) \subset S^*$. Hence f restricts to an automorphism of S^* . But S^* is noetherian since it is generated as a ring by a finite subset of R which is tightly F-noetherian. Hence the skew-Laurent subring $R^* = S^*[x, x^{-1}; f]$ is noetherian [B-W, Thm. 1.17]. Finally, it is easily seen that $R^* = S^*[x, x^{-1}; f]$ contains the given finite subset B of R . This proves Remark 4.6.

□

4.7 Proposition. *Let $R = S[x_1, (x_1)^{-1}; f_1][x_2, (x_2)^{-1}; f_2] \dots [x_n, (x_n)^{-1}; f_n]$ be an iterated skew-Laurent ring extension such that*

- (i) *for each $j > i$, $(f_j)(x_i) = q_{ji}x_i$ for some central unit q_{ji} in S , and*
- (ii) *each f_i is the identity on S .*

If S is F-noetherian, then so is R .

Proof. The proof is quite similar to the proof of Theorem 4.3 (with no $d_i s$) as follows. Briefly, for each finite subset B of R , we collect all “relevant” coefficients in S from writing each element of B as a polynomial in x and x^{-1} . To such finite coefficients in S , we add all q_{ji} and their inverses. The resulting finite subset of S is contained in a noetherian subring S' of S . Now $R' = S'[x_1, (x_1)^{-1}; f_1][x_2, (x_2)^{-1}; f_2] \dots [x_n, (x_n)^{-1}; f_n]$ is a well-defined iterated skew-Laurent ring extension. Moreover, R' is noetherian by [B-W, Thm. 1.17] and R' contains the given finite set B . Hence R is F-noetherian. This proves Proposition 4.7. □

5 Skew Quantum Ring Extensions

5.1 Definition. Following Passman in [P, p. 180], a ring extension R of a ring S is called an almost centralizing extension of S if $R = S[x_1, x_2, \dots, x_n]$ is generated as a ring over S by finitely many elements x_1, x_2, \dots, x_n such that

- (i) each x_i commutes with each element of S , and
- (ii) for all i, j , $[x_j, x_i] \in S + Sx_1 + \dots + Sx_n$.

However, in [M-R, 8.6.6], the definition of an almost centralizing extension is more general in the sense that R is generated as a ring over S by finitely many elements $\{x_1, x_2, \dots, x_n\}$ where

- (i) $sx_i - x_is \in S$ for each $s \in S$, and each i , and
- (ii) for all i, j , $[x_j, x_i] \in S + Sx_1 + \dots + Sx_n$.

But we shall stick to the first definition by Passman.

5.2 Theorem. *Let $R = S[x_1, x_2, \dots, x_n]$ be an almost centralizing extension of a ring S in the sense that R is generated as a ring over S by finitely many elements $\{x_1, x_2, \dots, x_n\}$ such that*

- (i) *each x_i commutes with the elements of S .*
- (ii) *for all i, j , $[x_j, x_i] \in S + Sx_1 + \dots + Sx_n$.*

If S is F -noetherian, then so is R . Thus, all iterated almost centralizing extensions of an F -noetherian ring are F -noetherian.

More generally, condition (ii) can be generalized to the following condition.

(ii)' For all $j > i$, there exist units $q_{ji} \in S$ such that

$$(x_j x_i - q_{ji} x_i x_j) \in S + Sx_1 + \dots + Sx_n$$

Theorem 5.2 is a special case of Theorem 5.4 case 2 part (ii) as shown below.

5.3 Lemma. *Let $R = S[x_1, x_2, \dots, x_n]$ be a finitely generated ring over a subring S such that*

- (i) *for all i , $S + Sx_i = x_i S + S$, and*
- (ii) *for all $j > i$, $(x_j x_i - q_{ji} x_i x_j) \in S + Sx_1 + \dots + Sx_n$ for some units q_{ji} in S .*

Then, if S is noetherian, then so is R .

Proof. The proof is almost verbatim as in the interesting proof of Thm. 6.14 by McConnell and Robson in [M-R, p. 29] since our added units q_{ji} of S do not hurt their proof. \square

In Theorem 5.4 below, we note that the noetherian part (5.4)(i) of case 3 is a slight generalization of Proposition I.8.17 in [B-G, p. 77] which is one of the key steps in proving that the quantum group $\mathcal{O}_q(G)$ is noetherian. One difference in our generalization is that we do not assume that each x_i commutes with the elements of S . We also note that the noetherian part 5.4(i) of case 2 below is a generalization of [L-R, Cor. 2.4] since we do not assume any PBW S -basis for R .

5.4 Theorem. *Let $R = S[x_1, x_2, \dots, x_n]$ be a finitely generated ring over a subring S satisfying the following two conditions.*

- (1) *for all i , $x_i S + S = S + S x_i$.*
- (2) *for all $j > i$, there exist units $q_{ji} \in S$ such that we have one of the following cases.*

Case 1: $(x_j x_i - q_{ji} x_i x_j) \in S[x_1, \dots, x_{j-1}] + S x_j$

Case 1': $(x_j x_i - q_{ji} x_i x_j) \in S x_i + S[x_{i+1}, \dots, x_n]$

Case 2: $(x_j x_i - q_{ji} x_i x_j) \in S + S x_1 + \dots + S x_n$

Case 3: $(x_j x_i - q_{ji} x_i x_j) = f + g$ where $f \in S + S x_1 + \dots + S x_n$ and if $i > 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $s x_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

Then in all cases, we have the following.

- (i) *If S is noetherian, then so is R .*
- (ii) *Suppose $S = A[z_1, \dots, z_m]$ is a finitely generated ring over a subring A such that each x_i commutes with the elements of A . Then, if S is F -noetherian, so is R .*
- (iii) *Under the additional assumption in (ii), if S is directed F -noetherian, then so is R .*

Proof.

- (i) The proof in case 3 is a simple modification of the interesting filtration method in [B-G, p. 77]. For convenience, we shall start with case 2.

Case 2: Here we have $x_j x_i - q_{ji} x_i x_j \in S + Sx_1 + \cdots + Sx_n$. So we can take the standard filtration $A_d := (S + Sx_1 + \cdots + Sx_n)^d$ for $d \geq 1$, while $A_0 := S$. Let $y_i = x_i + A_0 \in A_1$. Then, since $S + Sx_1 + \cdots + Sx_n \in A_1$, it follows that $\text{gr}(R) = S[y_1, y_2, \dots, y_n]$ such that $y_j y_i = q_{ji} y_i y_j$ for all $j > i$. Moreover, $y_i S = S y_i$ for all i since for all i , $x_i S + S = S + Sx_i$. Hence, since S is noetherian, we can apply Lemma 5.3 to obtain that R is noetherian.

Case 1: Here we have $x_j x_i - q_{ji} x_i x_j = f_{ji} + s x_j \in S[x_1, \dots, x_{j-1}] + Sx_j$ for some $s \in S$. So we choose the degree of every element x_i by the formula $d(x_i) = d_i = N^i$ where N is the maximum usual degree among all f_{ji} , and we choose $d(s) = 0$ for all $s \in S$. Next we define a non-negative filtration on R with $A_0 = S$ and for $d \geq 1$, A_d is the set of all finite sums of products $t_1 t_2 \dots t_r$ where each $t_i \in S \cup \{x_1, x_2, \dots, x_n\}$ such that

$$d(t_1) + d(t_2) + \cdots + d(t_r) \leq d.$$

Then $N \cdot d_{j-1} \leq d_i + d_j$ because it says $N \cdot N^{j-1} \leq N^i + N^j$. Hence each $f_{ji} \in A_{d_i + d_j - 1}$. Moreover, $d_j \leq d_i + d_j$. Consequently, $f_{ji} + s x_j \in A_{d_i + d_j - 1}$ for all $s \in S$. Again, as above, we end up with $\text{gr}(R) = S[y_1, \dots, y_n]$ such that $y_j y_i = q_{ji} y_i y_j$ for all $j > i$, and $y_i S + S = S y_i + S$ for all i . Since S is also noetherian, we can apply Lemma 5.3 to obtain that R is noetherian.

Case 1': Here we have $x_j x_i - q_{ji} x_i x_j = s x_i + f_{ji}$ for some $s \in S$ and some $f_{ji} \in S[x_{i+1}, \dots, x_n]$. In fact, case 1' is equivalent to case 1 by reversing the order of elements. That is, using the permutation that sends i to $n - i + 1$ to get the ordering $\{x_n, x_{n-1}, \dots, x_1\}$. Or we can modify the proof for case 1 by choosing $d_i = N^{n-i}$ where N is the maximum of all usual degrees of f_{ji} .

Case 3: Our proof of for case 3 would be a simple modification of the proof of Prop. I.8.17 in [B-G, p. 77] where it is assumed versus our generalization that each x_i commutes with the elements of k , and it is assumed that our $f = 0$ and that $y_j y_i - q_{ji} y_i y_j = 0$ for $i = 1$. Note that the authors in [B-G] used $i > j$ versus our notation $j > i$. Their trick was to assign a degree d_i for each x_i that leads to filtration such that $\text{gr}(R)$, in their case, is generated as an S -algebra by the elements y_1, y_2, \dots, y_n satisfying

$y_j y_i = y_i y_j$ for all $j > i$. In fact their chosen degrees, which we shall keep, is $d(x_i) = d_i = 2^n - 2^{n-i}$, and each $d(s) = 0$ for all $s \in S$. In particular, their non-negative filtration starts with $A_0 = S$. However, we shall slightly modify the definition of A_d , for $d \geq 1$, to be all finite sums of words $t_1 t_2 \dots t_r$ where each $t_i \in S \cup \{x_1, x_2, \dots, x_n\}$ such that

$$d(t_1) + d(t_2) + \dots + d(t_r) \leq d$$

This gives a non-negative filtration of R . The choice $d(x_i) = 2^n - 2^{n-i}$ makes our all the quadratic elements $s x_a x_b$ (appearing in g in case $i > 1$) (so a or b is at most $i - 1$) have the property that $d_a + d_b \leq d_i + d_j$ as shown in [B-G, p. 77]. Now we check our additional terms. In case $i = 1$, the quadratic terms in g have the property $d_1 + d_{j-1} \leq d_1 + d_j$. Hence, for all $j > i$, f_{ji} or $f \in A_{d_i + d_{j-1}}$. Moreover, $S + Sx_1 + \dots + Sx_n \subset A_{d_i + d_{j-1}}$ for all $j > i$ because One can easily show that $d(x_n) = d_n = 2^n - 1 < d(x_1) + d(x_2) = (2^n - 2^{n-1}) + (2^n - 2^{n-2}) - 1$ because upon simplification, we need $3(2^{2n-3}) - 2^n + 1 > 0$ for all $n \geq 2$ which is true since $\frac{3}{8}(x^2) - x + 1 > 0$ for all $x \geq 2$. Hence $\text{gr}(R) = [y_1, y_2, \dots, y_n]$ such that $y_j y_i = q_{ji} y_i y_j$ for all $j > i$ and $y_i S + S = S + S y_i$. Since S is noetherian, we can apply Lemma 5.3 to obtain that R is noetherian.

- (ii) Since each x_i are not assumed to commute with the elements of S , then every element of the finitely generated ring $R = S[x_1, x_2, \dots, x_n]$ can be written as a finite sum of words t_1, t_2, \dots, t_k where each $t_i \in S \cup \{x_1, x_2, \dots, x_n\}$. Such representation is not necessarily unique. But we use the *axiom of choice* to make a fixed choice of representations for all elements of R . Let $F = \{f_1, \dots, f_m\}$ be any finite subset of R . For all i , let $C(f_i)$ be the elements of S appearing in the representation of f_i . Similarly, for all $j > i$, let C_{ji} be the elements of S appearing in the representation of f_{ji} depending on our case. Recall

$$x_j x_i - q_{ji} x_i x_j = f_{ji}$$

Let F' be the union of the following finite sets: $C(f_i)$ with all C_{ji} with all q_{ji} . We need to add more constants from S to F'' arising from the given condition: for all i , $x_i S + S = S + S x_i$. Thus each $x_i z_j \in S x_i + S$ and each $z_j x_i \in S x_i + S$. Hence

$$x_i z_j = b_{ij} x_i + c_{ij} \text{ and } z_j x_i = x_i d_{ij} + e_{ij}$$

where all $b_{ij}, c_{ij}, d_{ij}, e_{ij} \in S$.

Now we enlarge the finite set F' by adding all $b_{ij}, c_{ij}, d_{ij}, e_{ij}$, and let $X(F)$ be the resulting set. Note that $X(F)$ is a *finite* subset of S because we have finitely many z_i in

$S = A[z_1, z_2, \dots, z_m]$. For (iii) in particular, we shall need the axiom of choice again to write all elements of $A[z_1, \dots, z_m]$ as words like $a_1 z_2 z_3 a_2 z_1 a_3$. With this in mind, let $X(F(A))$ be the set of all coefficients in A appearing in the representations of all $x \in X(F)$ as words in $A[z_1, \dots, z_m]$. Since S is F-noetherian, $X(F)(A)$ is *contained* in a noetherian subring A^* of S . Finally, let

$$S^* = A^*[z_1, \dots, z_m] \text{ and let } R^* = S^*[x_1, \dots, x_n]$$

To complete the proof, we shall need Lemma 5.5 proven below which shows that condition (1) is still satisfied. That is, for all i , $x_i S^* + S^* = S^* + S^* x_i$. We also have condition (2) is satisfied. Namely, for all $j > i$, there exist units $q_{ji} \in S^*$ such that all “coefficients” of S appearing in all the equations $x_j x_i - q_{ji} x_i x_j = f_{ji}$ are in S^* (where f_{ji} depends on our four cases). Hence $R^* = S^*[x_1, x_2, \dots, x_n]$ is noetherian by part (i) and thereby proving (ii).

- (iii) Now we are assuming that S is directed F-noetherian. Our proof in (ii) with the axiom of choice (used twice), shows that for finite subsets $F_1 \subset F_2$ in $R = S[x_1, x_2, \dots, x_n]$, we have constructed the corresponding finite sets in S and then in A (via where $S = A[z_1, \dots, z_m]$) such $X(F_1)(A) \subset X(F_2)(A)$. These two finite sets are contained respectively in noetherian subrings $S_1^* \subset S_2^*$ since S is directed F-noetherian. Thus $S_1^*[x_1, x_2, \dots, x_n] \subset S_2^*[x_1, x_2, \dots, x_n]$. Hence R is directed F-noetherian. This proves Theorem 5.4 modulo the proof of Lemma 5.5 shown next.

□

5.5 Lemma. *Let $R = S[x_1, x_2, \dots, x_n]$ be a finitely generated ring over its subring S and suppose $S = A[z_1, \dots, z_m]$ where each x_i commutes with the elements of A , and for all i ,*

$$x_i S + S = S + S x_i$$

. In particular, we have $x_i z_j = b_{ij} x_i + c_{ij}$ and $z_j x_i = x_i d_{ij} + e_{ij}$ for some $b_{ij}, c_{ij}, d_{ij}, e_{ij} \in S$. Let A^ be a subring of A and let $S^* = A^*[z_1, \dots, z_m]$. If*

S^* contains all $b_{ij}, c_{ij}, d_{ij}, e_{ij}$ defined above, then for all i , we have

$$x_i S^* + S^* = S^* + S^* x_i$$

Proof. The elements of $S^* = A^*[z_1, \dots, z_m]$ (which is a finitely generated ring over A^*) can be written as a finite sum of words y_1, y_2, \dots, y_k (say of length k) where each $y_i \in A^* \cup \{z_1, z_2, \dots, z_m\}$. To check each $x_i S^* + S^* = S^* + S^* x_i$, we shall first prove the left-hand side, namely that each $x_i S^* \subset S^* + S^* x_i$ by induction on the length of words in S^* (while the right-hand side follows similarly). Recall that

$$x_i z_j = b_{ij} \cdot x_i + c_{ij} \text{ and } z_j \cdot x_i = x_i \cdot d_{ij} + e_{ij}$$

where $b_{ij}, c_{ij}, d_{ij}, e_{ij} \in S^*$.

By induction, suppose the words in S^* of length k satisfy the desired (left) property that each $x_i S^* \subset S^* + S^* x_i$. So let us take a word w of length $k + 1$. Since $S^* = A^*[z_1, \dots, z_m]$, the word w will have one of the following 4 forms.

$$a^* s^*, s^* a^*, z_j s^*, s^* z_j$$

where $a^* \in A^*$ and $\text{length}(s^*) = k$

Recall that each x_i commutes with elements of A , $A^* \subset A$, and, by induction, each $x_i s^* \subset S^* + S^* x_i$.

For form 1, each $x_i(a^* s^*) = a^* x_i s^* \in a^*(S^* x_i + S^*) \subset S^* x_i + S^*$. For form 2, each $x_i(s^* a^*) = (x_i s^*) a^* \in (S^* x_i + S^*) a^* = S^* a^* x_i + S^* a^* \subset S^* x_i + S^*$. For form 3, each $x_i(s^* z_j) \in (S^* x_i + S^*) z_j = S^*(b_{ij} \cdot x_i + c_{ij}) + S^* z_j \subset S^* x_i + S^*$ because S^* contains each b_{ij} and c_{ij} . For form 4, each $x_i(z_j s^*) = x_i \cdot z_j s^* = (b_{ij} x_i + c_{ij}) s^* \in b_{ij}(S^* x_i + S^*) + c_{ij} s^* \subset S^* x_i + S^*$ because S^* contains each b_{ij} and c_{ij} .

Similarly, we can show that each $s^* x_i \subset S^* + x_i S^*$ by using the equations $z_j x_i = x_i d_{ij} + e_{ij}$ and S^* contains each d_{ij} and e_{ij} . This proves Lemma 5.5. \square

We remind the reader that the proof of Lemma 5.5 completes the proof of Theorem 5.4.

5.6 Corollary. *Let $R = S[x_1, x_2, \dots, x_n]$ be a finitely generated ring over a subring S where each x_i commutes with the elements of S , and for all $j > i$, there exist units $q_{ji} \in S$ such that we have one of the following cases.*

Case 1: $(x_j x_i - q_{ji} x_i x_j) \in S[x_1, \dots, x_{j-1}] + S \cdot x_j$

Case 1': $(x_j x_i - q_{ji} x_i x_j) \in S x_i + S[x_{i+1}, \dots, x_n]$

Case 2: $(x_j x_i - q_{ji} x_i x_j) \in S + Sx_1 + \cdots + Sx_n$

Case 3: $(x_j x_i - q_{ji} x_i x_j) = f + g$ where $f \in S + Sx_1 + \cdots + Sx_n$ and if $i > 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

Then in all such cases, we have the following.

- (i) If S is noetherian, then so is R .
- (ii) If S is F -noetherian, then so is R .
- (iii) If S is directed F -noetherian, then so is R .

5.7 Corollary. Let $R = S[x_1; f_1, d_1][x_2; f_2, d_2] \cdots [x_n; f_n, d_n]$ be an iterated Ore extension over a ring S such that

- (i) for all $j > i$, $f_j(x_i) = q_{ji}(x_i)$ for some unit $q_{ji} \in S$, and
- (ii) for all i , $x_i S + S = S + Sx_i$ and $S = A[z_1, \dots, z_m]$ (is a finitely generated ring over a subring A) such that each x_i commutes with the elements of A .

Then, if S is F -noetherian, then so is R .

Proof. Here we have $(x_j x_i - q_{ji} x_i x_j) \in k[x_1, \dots, x_{j-1}]$. So we can apply Theorem 5.4 case 1 to obtain our result.. \square

5.8 Theorem. Let $G = k[t_1, \dots, t_m][x_1, \dots, x_n] = k[t_1, \dots, t_m, x_1, \dots, x_n]$ be a finitely generated ring over a subring k such that

- (1) For all $j > i$, there exist units $p_{ji} \in k$ such that

$$t_j t_i - p_{ji} t_i t_j \in k[t_1, \dots, t_{j-1}] + k t_j$$

- (2) For all $j > i$, there exist units $q_{ji} \in k$ such that

Case 1: $(x_j x_i - q_{ji} x_i x_j) \in k[x_1, \dots, x_{j-1}] + kx_j$

Case 2: $(x_j x_i - q_{ji} x_i x_j) \in k + kx_1 + \cdots + kx_n$

Case 3: $(x_j x_i - q_{ji} x_i x_j) = f + g$ where $f \in S + Sx_1 + \cdots + Sx_n$ and if $i > 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

(3) For all j and i , there exist units $c_{ji} \in k$ such that

$$t_i x_j - c_{ji} x_j t_i \in k + kt_1 + \cdots + kt_u + kx_1 + \cdots + kx_v$$

where in case 1, $(u, v) = (i, j-1)$ or $(i-1, j)$; in case 2, $(u, v) = (i, n)$; and in case 3, $(u, v) = (i-1, j)$.

(4) for all i , $kt_i + k = k + t_i k$,

(5) for all j , $kx_j + k = k + x_j k$, and

(6) $S = A[z_1, \dots, z_m]$ is a finitely generated ring over a subring A such that each x_i and each t_j commutes with the elements of A .

Then in all cases, we have the following.

- (i) If k is noetherian, then so is G .
- (ii) If k is F -noetherian, then so is G .
- (iii) If k is directed F -noetherian, then so is G .

Proof.

- (i) The proof is very similar to the proof of Theorem 5.4 (i) with the following modifications.

Case 1: We choose $\deg(t_i) = N^i$ where N is the maximum usual degree of all polynomials f_{ji} in the relations $t_j t_i - p_{ji} t_i t_j = f_{ji} \in k[t_1, \dots, t_{j-1}] + k.t_j$. We also choose $\deg(x_i) = M^i$ where M is the maximum usual degree of all polynomials h_{ji} in the relations $x_j x_i - q_{ji} x_i x_j = h_{ji} \in k[x_1, \dots, x_{j-1}] + k.x_j$. Then the strategy of the proof in Theorem 5.4 (i) works for the terms containing $t_j t_i - p_{ji} t_i t_j$ and the terms containing $x_j x_i - q_{ji} x_i x_j$. So we still need to check the relations containing $t_i x_j - c_{ji} x_j t_i$. For this, assuming $(u, v) = (i, j-1)$, all we need is $N^u + M^v = N^i + M^{j-1} < N^i + M^j$ which is true. Similarly, if $(u, v) = (i-1, j)$,

Case 2: We go as in case 1 except that we choose $\deg(x_i) = 1$. Then the strategy of the proof in Theorem 5.4 (i) works for the terms containing $t_j t_i - p_{ji} t_i t_j$ and the terms containing $x_j x_i - q_{ji} x_i x_j$. So we still need to check the relations containing $t_i x_j - c_{ji} x_j t_i$. For this, all we need is $N^u + 1 = N^{i-1} + 1 < N^i + 1$ which is true.

Case 3: We go as in case 1 except that we choose $\deg(x_i) = 2^n - 2^{n-i}$ as done in the proof of Theorem 5.4 (i). Then the strategy of the proof in that theorem works for the terms containing $t_j t_i - p_{ji} t_i t_j$ and the terms containing $x_j x_i - q_{ji} x_i x_j$. So we still need to check the relations containing $t_i x_j - c_{ji} x_j t_i$. For this, all we need is $N^u + 2^n - 2^{n-v} = N^{i-1} + 2^n - 2^{n-j} < N^i + 2^n - 2^{n-j}$ which is true.

The proofs of part (ii) and (iii) are very similar to the proof of Theorem 5.4 (ii) and (iii) and will be left to the reader. This proves Theorem 5.8. \square

5.9 Theorem. *Let $G = k[t_1, \dots, t_m][x_1, \dots, x_n] = k[t_1, \dots, t_m, x_1, \dots, x_n]$ be a finitely generated ring over a subring k such that*

(1) *For all $j > i$, there exist units $p_{ji} \in k$ such that*

$$t_j t_i - p_{ji} t_i t_j \in k + kt_1 + \dots + kt_m$$

(2) *For all $j > i$, there exist units $q_{ji} \in k$ such that*

$$\textbf{Case 1: } (x_j x_i - q_{ji} x_i x_j) \in k[x_1, \dots, x_{j-1}] + k.x_j$$

$$\textbf{Case 2: } (x_j x_i - q_{ji} x_i x_j) \in k + kx_1 + \dots + kx_n$$

(3) *For all j and i , there exist units $c_{ji} \in k$ such that*

$$t_i x_j - c_{ji} x_j t_i \in k + kt_1 + \dots + kt_u + kx_1 + \dots + kx_v$$

where in case 1, $(u, v) = (i, j-1)$ or $(i-1, j)$; while in case 2, $(u, v) = (i, n)$.

(4) *for all i , $kt_i + k = k + t_i k$,*

(5) *for all j , $kx_j + k = k + x_j k$, and*

(6) *suppose $S = A[z_1, \dots, z_m]$ is a finitely generated ring over a subring A , such that each x_i and each t_j commutes with the elements of A .*

Then in all cases, we have the following.

(i) *If k is noetherian, then so is G .*

(ii) *If k is F -noetherian, then so is G .*

(iii) *If k is directed F -noetherian, then so is G .*

The proof is very similar to the proof of Theorem 5.8 with the following modifications on degrees where for all i, j , we let $\deg(t_i) = 1$ and $\deg(x_j) = 1$ as in the proof of Theorem 5.8.

5.10 Remark. If a ring R has a non-negative filtration whose associated graded ring $\text{gr}(R)$ is F-noetherian, the ring R may fail to be F-noetherian. The failure may occur even if $\text{gr}(R)$ is commutative.

Example. Let $S = \mathbb{Z}[x_0, x_1, \dots, x_n, \dots]$ be the commutative ring in infinitely many variables. Let $R = S[y; d]$ be the differential Ore extension where d is the derivation of S such that $d(x_i) = x_{i+1}$. So $yx_i - x_iy = x_{i+1}$ for all i . Then R is not noetherian because $R = S[t; d]$ is a free S -module and S is not noetherian. Moreover, $\{x_0, t\}$ with 1 generate R as a ring. Hence R is not F-noetherian. We make the standard filtration on R induced by $\text{degree}(y) = 2$ and $\text{degree}(x_i) = 1$ for all i . More precisely, $A_0 = \mathbb{Z}$ and $A_m = (Zy + Zx_0 + Zx_1 + \dots + Zx_n + \dots)^m$ for $m \geq 1$. Let $y^* = y + A_0$, and let $x_i^* = x_i + A_0$. Since $yx_i - x_iy = x_{i+1}$ for all i , It follows that $y^*.x_i^* = x_i^*.y^*$ for all i and all x_i^* commute. Hence $\text{gr}(R)$ is commutative so $\text{gr}(R)$ is F-noetherian even though R is not F-noetherian.

Finally, we show another filtration on R such that each A_m is finite dimensional and the associated graded ring $\text{gr}(R)$ is still commutative. We let $d(x_i) = i$ and $d(y) = 2$, and modify A_m accordingly to be the \mathbb{Z} finite sum of monomials $t_{i_1}t_{i_2}\dots t_{i_k}$ where each $t_i \in \{y, x_0, x_1, \dots, x_n, \dots\}$ such that

$$d(t_{i_1}) + d(t_{i_2}) + \dots + d(t_{i_k}) \leq m.$$

Then each A_m is evidently finite dimensional, and the associated graded ring $\text{gr}(R)$ is still commutative even though R is not F-noetherian.

6 Applications and Examples

Throughout this section, we shall say that x and y almost commute in a ring R if $xy = qyx$ for some unit $q \in R$. In this case, we also say that q is the supporting constant of xy .

For the convenience of the reader, we recall Corollary 5.6.

Corollary 5.6 Let $R = S[x_1, x_2, \dots, x_n]$ be a finitely generated ring over a subring S where each x_i commutes with the elements of S , and for all $j > i$, there exist units $q_{ji} \in S$ such that we have one of the following cases.

Case 1: $x_j x_i - q_{ji} x_i x_j \in S[x_1, \dots, x_{j-1}] + S.x_j$

Case 1': $(x_j x_i - q_{ji} x_i x_j) \in Sx_i + S[x_{i+1}, \dots, x_n]$

Case 2: $x_j x_i - q_{ji} x_i x_j \in S + Sx_1 + \dots + Sx_n$

Case 3: $(x_j x_i - q_{ji} x_i x_j) = f + g$ where $f \in S + Sx_1 + \dots + Sx_n$ and if $i > 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and either a or b is at most $i - 1$; however if $i = 1$, g is a finite sum of quadratic monomials $sx_a x_b$ where $s \in S$ and $a = 1$ and $b < j$ or vice versa ($b = 1$ and $a < j$).

Then

(i) If S is noetherian, then so is R .

(ii) If S is F-noetherian, then so is R .

6.1 Example. Consider the following k -algebra R over a ring k in the variables x, y, z with relations

$$\begin{aligned} w(x, y, z) &= 0 \\ xy &= q_1 yx + f(y, z) + p_1 x \\ xz &= q_2 zx + g(y, z) + p_2 x \\ zy + yz &= h(z) + p_3 y \end{aligned}$$

for some polynomials, $f(y, z), g(y, z) \in k[y, z]$, $w(x, y, z) \in k[x, y, z]$, each $p_i \in k$, and each q_i is a unit of k .

Then if k is noetherian or k is F-noetherian, so is R . To see this, let R^+ be the k -algebra with the last three given relations. Then R is a homomorphic image of R^+ . Since the noetherian and F-noetherian properties are preserved by homomorphic images, it suffices to work with R^+ . Then Corollary 5.6 applies to R^+ with the ordering shown in $R^+ = k[z, y, x]$.

6.2 Example. Let R be the k -algebra generated by the variables x_i, y_i, z_i , $1 \leq i \leq n$ such that any pair of variables almost commute (whose supporting constants are units in k) except for the following pairs where we have

$$x_i y_i + q_i y_i x_i = f_i + p_i x_i,$$

where $f_i \in k[y_1, \dots, y_n, z_1, \dots, z_n, x_1, \dots, x_{i-1}]$ for some units $q_i \in k$, some $p_i \in k$. Then if k is noetherian or k is F-noetherian, so is R . This follows from Corollary 5.6 with the ordering shown in

$$R = k[y_1, \dots, y_n, z_1, \dots, z_n, x_1, \dots, x_n]$$

6.3 Example. *The Woronowicz algebra over an F -noetherian ring k*

See [W] or [L-R, p. 1215]. Let R be the k -algebra generated by x, y, z subject to the relations

$$\begin{aligned}xz - v^4zx &= (1 + v^2)x \\xy - v^2yx &= vz \\zy - v^4yz &= (1 + v^2)y\end{aligned}$$

where v is a unit of k (which is not a root of unity). Then if k is noetherian or k is F -noetherian, so is R . This follows from Corollary 5.6 (case 2).

6.4 Example. *A generalization of the multi-parametrized quantum Weyl algebra over an F -noetherian ring k .*

For the multi-parametrized *quantum Weyl algebra* over a field k , see [L-R, p. 1218]. Let R be the k -algebra generated by the variables x_i, y_i , $1 \leq i \leq n$ such that any pair of variables almost-commute (whose supporting constants are units in k) except for the pairs (x_i, y_i) for all i , we have the relations, for all i ,

$$y_i x_i - q_i x_i y_i = f_i + p_i y_i$$

and

$$\text{case 1: } f_i \in k[x_1, y_1, \dots, x_{i-1}, y_{i-1}, \mathbf{x}_i]$$

$$\text{case 2: } f_i \in k[\mathbf{x}_i, x_{i+1}, y_{i+1}, \dots, x_n, y_n]$$

for some units $q_i \in k$ and some elements $p_i \in k$.

Then if k is noetherian or k is F -noetherian, so is R . The first case follows from Corollary 5.6 with the ordering shown in

$$R = k[x_1, y_1, \dots, x_i, y_i, \dots, x_n, y_n]$$

The second case follows from 5.6 with ordering shown in

$$R = k[x_n, y_n, \dots, x_i, y_i, \dots, x_1, y_1]$$

6.5 Example. *A generalization of the Hayashi algebra over an F -noetherian ring k .*

For the Hayashi algebra over a field k , see [H] or [L-R, p. 1217]. Let R be the k -algebra generated by the variables x_i, y_i, z_i , $1 \leq i \leq n$ such that any pair of variables almost-commute (whose supporting constants are units in k) except for the pairs (x_i, z_i) where we have the relations

$$(z_i x_i + q x_i z_i) y_i = 1 = y_i (z_i x_i + q x_i z_i)$$

for some unit $q \in S$. Observe that the above relation implies that

$$(y_i)^{-1} = z_i x_i + q x_i z_i.$$

Then if k is noetherian or k is F-noetherian, so is R . This follows from Corollary 5.6 by considering

$$R = k[(y_1)^{\pm 1}, \dots, (y_n)^{\pm 1}, x_1, \dots, x_n, z_1, \dots, z_n].$$

(Note that if $xy = qyx$, then $xy^{-1} = qy^{-1}x$ if q commutes with x and y).

6.6 Example. *On some quadratic algebras.*

Let R be the k -algebra generated by the variables x_i with $1 \leq i \leq 5$ such that any pair of variables almost-commute (whose supporting constants are units in k) except for the pairs (x_5, x_4) where we have the relation

$$x_5 x_4 - q x_4 x_5 = p_1 x_5 x_3 + p_2 x_5 x_2 + p_3 (x_3)^2$$

for some unit $q \in k$ and some constants $p_i \in k$ for $i = 1, 2, 3$. Now we can apply Theorem 5.4 to see that if k is noetherian or F-noetherian, then so is R .

6.7 Example. *The Quantum enveloping algebra of $U_q(sl(2, k))$ over an F-noetherian ring k .*

$U_q(sl(2, k))$ is defined as the k -algebra generated by x, y, z, z^{-1} with relations

$$\begin{aligned} xz &= q^{-2}zx \\ yz &= q^2zy \\ xy - yx &= (q - q^{-1})^{-1}(z - z^{-1}) \end{aligned}$$

where $q \in k$ and $q \neq \pm 1$. Then if k is F-noetherian, so is $U_q(sl(2, k))$.

We can see this in many ways. For example, we may apply Theorem 5.4 on $U_q(sl(2, k)) = S[x, y]$ where $S = k[z, z^{-1}]$. More precisely, since x and y are in the centralizer of k , and $xz = q^{-2}zx$, it follows that $xS + S = S + Sx$. Similarly, $yS + S = S + Sy$. Here we can apply Theorem 5.4 to see that if k is F-noetherian, so is $U_q(sl(2, k))$. For another proof, see (6.8) below.

6.8. We can also prove 6.6 by applying Prop. 3.9 part (ii) on $U_q(sl(2, k)) = k[x, y, z]Z^{-1}$ as a localization with respect to $Z = \{z^i \text{ with } i \in \mathbb{N}\}$

More precisely, since $xz = q^{-2}zx$, $yz = q^2zy$, $q \in k$ and the centralizer of k contains $Z \cup \{x, y, z\}$, it follows that Z satisfies all hypotheses of Proposition 3.9 part(ii) with the notation $R = k[x, y, z]$ and $Q = \{z\}$. Thus, if k is F-noetherian, so is $U_q(sl(2, k))$.

6.9 Example. *A generalization of $U_q(sl(2))(k)$.*

Let U be the k -algebra generated by $\{x, y, z, z^{-1}\}$ with the relations

$$\begin{aligned}xz &= q_1zx, \\yz &= q_2zy, \text{ and} \\xy - yx &= f(z, z^{-1}, x) + q_3y,\end{aligned}$$

where $f(z, z^{-1}, x) \in k[z, z^{-1}, x]$ and each q_i is a unit of k . Then, verbatim, as in the proof (6.6), we can apply Theorem 5.4 on $U = S[x, y]$ where $S = k[z, z^{-1}]$ to see that, if k is noetherian or F-noetherian, so is U .

6.10 Example. *Witten deformation of $U_q(sl(2))$ over an F-noetherian ring.*

E. Witten (see [L-R], p. 1217) introduced and studied a 7-parameter deformation of the universal enveloping algebra $U(sl(2, k))$ depending on a 7-tuple of parameters $\xi = (\xi_1, \dots, \xi_7)$ and subject to the relations

$$\begin{aligned}xz - \xi_1zx &= \xi_2x, \\zy - \xi_3yz &= \xi_4y, \text{ and} \\yx - \xi_5xy &= \xi_6z^2 + \xi_7z\end{aligned}$$

In the usual case where k is a field, it is assumed that $\xi_1, \xi_2, \xi_3 \neq 0$.

In our general case we assume that ξ_1, ξ_2, ξ_3 are units in k and the resulting algebra is denoted by $W(\xi)(k)$. If we apply Corollary 5.6 with the ordering shown in $W(\xi)(k) = k[x, z, y]$ or $W(\xi)(k) = k[z, x, y]$, we see that if k is noetherian or F-noetherian, so is $W(\xi)(k)$.

7 Some Quantum Groups over F-noetherian Rings.

Recall that we have discussed in (6.7) -(6.10) some variants of the Quantum enveloping algebra of $U_q(sl(2, k))$ over an F-noetherian ring k .

In this last section, we give many examples of some quantum groups over F-noetherian rings.

7.1. For convenience, a skew quantum *fully triangular* extension over k shall mean a k -algebra $k[t_1, \dots, t_m]$ generated by the variables $\{t_1, \dots, t_m\}$ such that for all $j > i$, there exist units $p_{ji} \in k$ such that

$$t_j t_i - p_{ji} t_i t_j \in k[t_1, \dots, t_{j-1}] + k \cdot t_j.$$

7.2. A generalization of the quantum group $\mathcal{O}_q(M_n(k))$ and its multi-parameter version.

Recall that the quantum group $\mathcal{O}_q(M(n, k))$ is the k -algebra generated by all variables x_{ij} with $1 \leq i, j \leq n$ such that each 2×2 square matrix of four generators at the positions $(i, j), (i, j'), (i', j), (i', j')$ where $i < i'$ and $j < j'$ generate a copy of the well-known quantum group $\mathcal{O}_q(M(2, k))$ of 2×2 matrices [B-G, I.2.2].

Let $G(\mathcal{O}_q(M_n(k))) = k[t_1, \dots, t_m, x_{11}, \dots, x_{nn}]$ be the k -algebra with the evident variables such that

- (i) $k[x_{11}, \dots, x_{nn}] = \mathcal{O}_q(M_n(k))$ (with q being a central unit of k) is the k -algebra generated by the variables x_{ij} with $1 \leq i, j \leq n$ and with the standard relations of the quantum group $\mathcal{O}_q(M_n(k))$ of $n \times n$ matrices. See [B-K, p. 16].

- (ii) For all $j > i$, there exist units $p_{ji} \in k$ such that

Case 1: $t_j t_i - p_{ji} t_i t_j \in k[t_1, \dots, t_{j-1}] + k t_j$

Case 2: $t_j t_i - p_{ji} t_i t_j \in k + k t_1 + \dots + k t_m$

- (iii) Let $\{x_1, \dots, x_{n^2}\}$ be the lexicographic ordering of $\{x_{11}, \dots, x_{nn}\}$. For all j and i , we also assume there exist units c_{ji} in k such that

$$t_i x_j - c_{ji} x_j t_i \in k + k t_1 + \dots + k t_u + k x_1 + \dots + k x_v$$

where in case 1, $(u, v) = (i, j-1)$ or $(i-1, j)$, while in case 2, $(u, v) = (m, n^2)$

Note that our given ring can be written as $G(\mathcal{O}_q(M_n(k))) = k[t_1, \dots, t_m][x_{11}, \dots, x_{nn}]$

Then, if k is noetherian, or F-noetherian, or directed F-noetherian, so is $G(\mathcal{O}_q(M_n(k)))$.

Proof. This follows immediately from Theorem 5.8 and Theorem 5.9 since the ring extension $\mathcal{O}_q(M_n(k))$ is fully triangular in the sense of (7.1). See [B-K, p. 16] for details.

Note. Similarly, we may consider a generalization of the multi-parameter version of $\mathcal{O}_q(M_n(k))$ where the parameters are central units of k . (See [B-K, p.16]). \square

7.3. A generalization of the quantum groups $\mathcal{O}_q(\mathrm{SL}_n(k))$ and $\mathcal{O}_q(\mathrm{GL}_n(k))$

Let $G(\mathcal{O}_q(M_n(k)))$ be as in (7.2) except that we replace condition (3) with the stronger condition

(3)' each t_i commutes with each x_j .

Let D_q be the quantum determinant of $\mathcal{O}_q(M_n(k))$, so D_q is central in $G(\mathcal{O}_q(M_n(k)))$ [B-K p. 17]. Hence D_q is also central in $G(\mathcal{O}_q(M_n(k)))$ by condition (3)' above. So we can form

$$\begin{aligned} G(\mathcal{O}_q(\mathrm{SL}_n(k))) &= G(\mathcal{O}_q(M_n(k)))/[D_q - 1] \\ G(\mathcal{O}_q(\mathrm{GL}_n(k))) &= G(\mathcal{O}_q(M_n(k)))[(D_q)^{-1}] \end{aligned}$$

If k is noetherian, or F-noetherian, or directed F-noetherian, then so is $G(\mathcal{O}_q(\mathrm{SL}_n(k)))$ and $G(\mathcal{O}_q(\mathrm{GL}_n(k)))$.

Proof. For $G(\mathcal{O}_q(\mathrm{SL}_n(k))) = G(\mathcal{O}_q(M_n(k)))/[D_q - 1]$, this follows immediately from (7.2) since the noetherian, the F-noetherian, and the directed F-noetherian properties are preserved under homomorphic images (3.5).

For $G(\mathcal{O}_q(\mathrm{GL}_n(k))) = G(\mathcal{O}_q(M_n(k)))[(D_q)^{-1}]$, this follows immediately from (7.2) since the noetherian, the F-noetherian, and the directed F-noetherian properties are preserved under localizations by a central subset (see 3.9). \square

7.4. A generalization of the quantum symplectic algebra $\mathcal{O}_q(\mathrm{SP}_n(k))$ over an F-noetherian ring k .

Recall that the algebra $\mathcal{O}_q(\mathrm{SP}_n(k))$ is the k -algebra generated by the variables x_i, y_j with $i, j = 1, \dots, n$ subject to the relations

$$\begin{aligned} y_j x_i &= q^{-1} x_i y_j, & y_j y_i &= q y_i y_j, & 1 \leq i < j \leq n \\ x_j x_i &= q^{-1} x_i x_j, & x_j y_i &= q y_i x_j, & 1 \leq i < j \leq n \\ x_i y_i - (q^2) y_i x_i &= (q^2 - 1) \sum_{t=1}^{i-1} q^{i-t} y_t x_t, & 1 \leq i \leq n \end{aligned}$$

Then $\mathcal{O}_q(\mathrm{SP}_n(k))$ is a triangular extension as in (7.1) via the ordering shown in $k[x_1, y_1, x_2, y_2, \dots, x_n, y_n]$.

Again, we can generalize $\mathcal{O}_q(\mathrm{SP}_n(k))$ to $G(\mathcal{O}_q(\mathrm{SP}_n(k)))$ exactly as in (7.2).

In conclusion, if k is noetherian or F-noetherian or directed F-noetherian, then so is $G(\mathcal{O}_q(\mathrm{SP}_n(k)))$.

7.5 Remark. Let S be a ring containing a field k of characteristic 0, and let q be a unit of k . Then, by extension of scalars, we may form the quantum enveloping algebra $U_q(L)(S)$ of a simple Lie algebra L from the k -algebra $U_q(L)(k)$ and we can form the quantum group $\mathcal{O}_q(G)(S)$ of a connected semi-simple algebraic group G from the k -algebra $\mathcal{O}_q(G)(k)$ (see [B-G, chapters 1.6, 1.7]). It is well-known that $U_q(L)(k)$ is noetherian [B-G, p. 55] and that $\mathcal{O}_q(G)(k)$ is also noetherian [B-G, p. 78].

Problem. I expect a positive answer to the problem whether if S is noetherian or F-noetherian or directed F-noetherian, then so are $U_q(L)(S)$ and $\mathcal{O}_q(G)(S)$.

8 Examples of F-noetherian Matricial Rings and F-noetherian Group Algebras.

In this last section, we give many matricial examples and many group algebras examples of F-noetherian rings where some examples are tightly F-noetherian and some are non-tightly F-noetherian.

8.1 Proposition. *Let $M_n(R)$ be the ring of $n \times n$ matrices over a ring R .*

- (i) **Examples of tightly F-noetherian rings.** *If $R = \mathbb{Z}$, then $M_n(R)$ is tightly F-noetherian. More generally, $M_n(R)$ is tightly F-noetherian if R a ring which is a module-finite ring extension of either \mathbb{Z} or R is a prime field \mathbb{Z}_p . (Interesting examples of such coefficient rings R in (iv) are the finite fields or the ring of algebraic integers of a number field (since it has a finite basis over \mathbb{Z}) [Ma, p. 30]).*
- (ii) **Example of an F-noetherian ring that is not tightly F-noetherian.** *$M_2(\mathbb{Z}[x])$ is noetherian but not tightly F-noetherian.*
- (iii) **Example of an F-noetherian ring that may not be tightly F-noetherian.** *If R is an F-noetherian ring, then so is $M_n(R)$.*

Proof.

- (i) Every subring A (with 1) of $M_n(\mathbb{Z})$ is noetherian because A is a finitely generated \mathbb{Z} -module. Hence $M_n(\mathbb{Z})$ is tightly F-noetherian.

In general, let A be any subring of $M_n(S)$. Then $\mathbb{Z} \subset A \subset M_n(S)$. Now $M_n(S)$ is evidently a finitely generated \mathbb{Z} -module and A is a \mathbb{Z} -submodule of $M_n(S)$. Hence A is noetherian. Hence $M_n(S)$ is tightly F-noetherian. The second case (with \mathbb{Z}_p instead of \mathbb{Z}) is very similar.

- (ii) The matrix ring $M_2(\mathbb{Z}[x])$ is noetherian since it is a finitely generated $\mathbb{Z}[x]$ -module and $\mathbb{Z}[x]$ is noetherian. But this ring is not tightly F-noetherian as follows. Take the following set X of 18 upper triangular 2×2 matrices listed by rows as:

$$X := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b \in \{0, 1, x\} \text{ and } c \in \{0, 1\} \right\}.$$

Then the subring generated by X is exactly the subring A of 2×2 matrices of the form

$$\begin{pmatrix} \mathbb{Z}[x] & \mathbb{Z}[x] \\ 0 & \mathbb{Z} \end{pmatrix}$$

with no relation among the entries. But $\mathbb{Z}[x]$ is not a noetherian \mathbb{Z} -module. So A is not noetherian [L 1, (1.22)]. Hence $M_2(\mathbb{Z}[x])$ is not tightly F-noetherian (while it is F-noetherian).

- (iii) Let A be a finite subset of $M_n(R)$. Then the finitely many entries of the elements of A are contained in a noetherian subring R' of R . Hence the subring $M_n(R')$ is noetherian (and containing the finite subset A). In general, such matricial rings may fail to be tightly F-noetherian (see Example (x) below). In general, $M_n(R)$ may fail to be tightly F-noetherian by (ii).

□

8.2. Examples from Group Algebras

1. *Example of a tightly F-noetherian ring.* Let $Z[G]$ be the group algebra of a locally finite group G over Z . Then $Z[G]$ is tightly F-noetherian. To see this, let X be a finite subset of $Z[G]$. Then X involves finitely many basis elements of G which generate a finite subgroup G' of G . Now $Z[G']$ is a finitely generated \mathbb{Z} -module. Hence the subring generated by X is noetherian being a sub \mathbb{Z} -module of $Z[G']$. Hence $Z[G]$ is tightly F-noetherian.
2. *Example of an F-noetherian rings which is not tightly F-noetherian.* Let K be a noetherian domain, and let G be a free group with at least

two generators. Note that K can be embedded in a division ring D [L 2, (10.23)]. Now let R be any division hull of the group K -algebra $K[G]$ which exists since $D[G]$ has a division hull where one such division hull is the Malcev-Neumann construction for an ordered group G over D because free groups are ordered ([L 1, (6.31), (14.24)] or [Passman 1]). Then R is F-Noetherian since it is a division ring. Now let G' be the free subgroup on two generators $\{a, b\}$. Then the subring $\mathbb{Z}[G']$ is not noetherian because, for example, the free algebra $\mathbb{Z}[X]$ with $|X| \geq 2$ is not noetherian as in [L 2, (1.31)]. Here $\mathbb{Z}[G'] = \mathbb{Z}[X, X^{-1}]$. Hence R (which is any division hull of $K[G]$) is not tightly F-noetherian although R is F-noetherian.

3. *Example of an F-noetherian rings which is not tightly F-noetherian.* A slight variation of the preceding example is when R is any division hull of the free k -algebra $K[X]$ with $|X| \geq 2$. (Note that $K[X]$ is a proper subring of $K[G]$ where G is the free group generated by X). Then R is F-noetherian. Moreover, if $\{a, b\}$ is a two-element subset of X , then as above, the free subring $\mathbb{Z}[a, b]$ is not noetherian. Hence R (which is any division hull of $K[X]$) is not tightly F-noetherian although R is F-noetherian.
4. *Example.* Let $K[G]$ be the group algebra of a locally finite group G over an F-noetherian ring K . (Recall that a group G is locally finite if every finitely generated subgroup is finite). (Interesting examples of G are the finitary symmetric/alternating groups on an infinite set). Then $K[G]$ is F-noetherian. To see this, let X be a finite subset of $K[G]$. Then we have the elements in X involve finitely many basis elements of G which generate a finite subgroup G' of X , and the elements in X involve finitely many elements of K which are contained in a noetherian subring K' of K , such that X is contained in $K'[G']$. Now $K'[G']$ is a finitely generated K' -module. Hence $K'[G']$ is a noetherian ring containing X . Hence $K[G]$ is F-noetherian.
5. *Example.* Let $K[G]$ be the group algebra of a polycyclic-by-finite group G (for example G is a finitely generated nilpotent group) over an F-noetherian ring K . Recall that a group G is polycyclic-by-finite if G has a polycyclic normal subgroup of finite index. Then $K[G]$ is F-noetherian as follows. Let X be a finite subset of $K[G]$. Then the coefficients of the elements of X form a finite subset of K which is contained in a noetherian subring K' of K . Now $K'[G]$ is a noetherian ring [J, p. 305] containing X . Hence $K[G]$ is F-noetherian.

The following Proposition is known; but in the absence of a reference, we give a proof.

8.3 Proposition. *The direct limits of noetherian rings are precisely the directed F-noetherian rings (as defined in section 3).*

Proof. . First note that the direct limit of noetherian rings are evidently the rings which are directed unions of noetherian subrings. Now every directed F-noetherian ring (as defined in section 3) is evidently a directed union of noetherian subrings.

Conversely, let R be a directed union of noetherian subrings. Then there exists a set S of noetherian subrings of R which is directed (every two members R_1, R_2 of S are contained in some member R_3 of S) and whose union is R . Now, for every 1-element set $\{a\}$, we can choose $R(a) \in S$ which contains a . Now suppose inductively that for some natural number $m \geq 1$ we have chosen, for every m -element subset A of R , an $R(A) \in S$ which contains A . For each $(m+1)$ -element set A , the directedness of S allows us to choose an $R(A) \in S$ which contains $R(B)$ for all of the finitely many proper subsets B of A . Thus induction on m completes the required construction. \square

9 Open Problems

In this paper, our examples of F-noetherian rings can be shown to be directed F-noetherian (being built on directed F-noetherian rings). We also know that F-noetherian rings are basic. So we pose the following two problems.

1. **Problem.** *Find an example of an F-noetherian ring which is not directed F-noetherian. Or equivalently, find an example of an F-noetherian ring which is not a direct limit of noetherian rings. See (8.3).*
2. **Problem.** *Find an example of a basic ring which is not F-noetherian.*
3. We briefly recall Problem 7.5. Let S be a ring containing a field k of characteristic 0, and let q be a unit of k . Then, by extension of scalars, we may form the quantum enveloping algebra $U_q(L)(S)$ of a simple Lie algebra L and we can form the quantum group $\mathcal{O}_q(G)(S)$ of a connected semi-simple algebraic group G . (see [B-G, chapters 1.6, 1.7]). I expect a positive answer to the question whether if S is noetherian

or F-noetherian or directed F-noetherian, then so are $U_q(L)(S)$ and $\mathcal{O}_q(G)(S)$.

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